GRID GRAPHS, GORENSTEIN POLYTOPES, AND DOMINO STACKINGS

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ABSTRACT. We examine domino tilings of rectangular boards, which are in natural bijection with perfect matchings of grid graphs. This leads to the study of their associated perfect matching polytopes, and we present some of their properties, in particular, when these polytopes are Gorenstein. We also introduce the notion of domino stackings and present some results and several open questions. Our techniques use results from graph theory, polyhedral geometry, and enumerative combinatorics.

1. INTRODUCTION

Our goal is to study, from a convex geometric point of view, domino tilings of rectangular boards. They are in natural bijection with perfect matchings of grid graphs. Namely, we consider their associated perfect matching polytopes and present some of their properties. In particular, we characterize when these polytopes are Gorenstein. (We will define all these notions shortly). We also introduce the notion of domino stackings and present some results and several open questions.

The $m \times n$ **grid graph** $\mathcal{G}(m,n)$ is defined with vertex set $\{(i,j) \in \mathbf{Z}^2 : 0 \leq i < n, \ 0 \leq j < m\}$ such that two vertices (i,j) and (i',j') are adjacent if and only if $|i-i'| + |j-j'| = 1$. The $m \times n$ **torus graph** $\mathcal{G}_T(m,n)$ consists of the same vertex and edge set as $\mathcal{G}(m,n)$ with the additional edges $\{(0,j),(n-1,j) : 0 \leq j < m\}$ and $\{(i,0),(i,m-1) : 0 \leq i < n\}$. We use the convention that $V(G)$ and $E(G)$ denote the vertex set and edge set of a graph G.

Given a graph G, $M \subseteq E(G)$ is a **perfect matching** if every vertex of G is incident with exactly one edge of M. A generalization of a perfect matching is the notion of a **magic labelling of sum** t, which is a function $E(G) \to \mathbf{Z}_{\geq 0}$ such that for each vertex v, the sum of the labels of the edges incident to v equals t. Perfect matchings are magic labellings of sum $t = 1$. A natural question is how many magic labellings of sum t a given graph G has. In this paper, we are interested in the case when $G = \mathcal{G}(m,n)$, and denote the number of magic labellings of sum t by $T(m,n,t)$. In particular, we can fix any two parameters, let the other vary to get a sequence, and encode this sequence in a generating function.

The **perfect matching polytope** $\mathcal{P}$ associated to a graph G is defined to be the convex hull in $\mathbf{R}^{E(G)}$ of the incidence vectors of all perfect matchings of G. There is a natural identification between points in $\mathcal{P}$ and weighted graphs. We denote by $\mathcal{P}(m,n)$ and $\mathcal{P}_T(m,n)$ the perfect matching polytopes of $\mathcal{G}(m,n)$ and $\mathcal{G}_T(m,n)$, respectively.

For an integral polytope $\mathcal{P}$ (i.e., the vertices of $\mathcal{P}$ have only integer coordinates), let $L_{\mathcal{P}}(t)$ be the function that counts the number of integer points in $t\mathcal{P} := \{tx : x \in \mathcal{P}\}$ for $t \in \mathbf{Z}_{>0}$. It was proved by Ehrhart [11] that $L_{\mathcal{P}}(t)$ agrees with a polynomial with constant term 1 for all positive integers.

See also [4] for a modern treatment of Ehrhart theory. It is easy to see (as we will show) that the magic labellings of sum t of $\mathcal{G}(m, n)$ correspond bijectively to the integer points of $t\mathcal{P}(m, n)$. It is a well-known result (see [21, Chapter 4.3]) that if $p(t)$ is a polynomial, then $\sum_{t \geq 0} p(t)z^t$ evaluates to a proper rational function (i.e., the degree of the numerator is strictly less than the degree of the denominator) of z as follows.

$$\sum_{t \geq 0} T(m, n, t)z^t = \sum_{t \geq 0} L_{\mathcal{P}(m,n)}(t)z^t = \frac{h(z)}{(1 - z)^{d+1}} =: \mathrm{Ehr}_{\mathcal{P}(m,n)}(z),$$

for a polynomial $h(z) = h_k z^k + h_{k-1} z^{k-1} + \cdots + h_0$ where $k \leq d$ and $h_k \neq 0$. We are interested in properties of the sequence $(h_0, h_1, \ldots, h_k)$, which we call the **Ehrhart h-vector** of $\mathcal{P}(m, n)$. For instance, Stanley showed (for general integral polytopes) that these numbers are nonnegative integers [20]. We characterize the values (m, n) for which $\mathcal{P}(m, n)$ is **Gorenstein**, that is, the Ehrhart h-vector is palindromic ($h_j = h_{k-j}$) [3, 5]. An equivalent formulation for the Gorenstein property is that there exists an integer k such that $L_{\mathcal{P}^\circ}(t) = L_{\mathcal{P}}(t - k)$ for all $t \geq k$ and $L_{\mathcal{P}^\circ}(t) = 0$ for $t < k$. Here $L_{\mathcal{P}^\circ}(t)$ denotes the number of interior integer points of $t\mathcal{P}$. In this case, we say that $\mathcal{P}$ is **Gorenstein of index k**.

To warm up, we show in Section 2 that the Gorenstein property holds for $\mathcal{P}_T(m, n)$ for some values. We can conclude this immediately from the following more general result.

Proposition 1. *Let G be a k-regular bipartite graph with $\#(V(G))$ even. Then the perfect matching polytope $\mathcal{P}$ of G is Gorenstein of index k. In particular, we have the functional identity $L_{\mathcal{P}^\circ}(t) = L_{\mathcal{P}}(t - k)$ for $t \geq k$.*

Corollary 2. *Assume $m \leq n$ and $n > 2$ is even.*

(1) $\mathcal{P}_T(1, 2)$ *is a point.*
(2) $\mathcal{P}_T(1, n)$ *and* $\mathcal{P}_T(2, 2)$ *are Gorenstein of index 2.*
(3) $\mathcal{P}_T(2, n)$ *is Gorenstein of index 3.*
(4) *If $m > 2$ is even, then $\mathcal{P}_T(m, n)$ is Gorenstein of index 4.*

It is worth noting that Tagami [24] has shown that these and $\mathcal{P}_T(2, 3)$ and $\mathcal{P}_T(2, 5)$ are all of the perfect matching polytopes of torus graphs which are Gorenstein; the latter two are Gorenstein of index 4 and 3, respectively.

For the non-torus grid graphs, we prove the following classification in Section 3, which is our first main result.

Theorem 3. *Assume $m \leq n$. The polytope $\mathcal{P} = \mathcal{P}(m, n)$ is Gorenstein of index k if and only if one of the following holds:*

(1) $m = 1$ *and n is even, in which case $\mathcal{P}$ is a point,*
(2) $m = 2$, *in which case $k = 2$ if $n = 2$, and $k = 3$ for $n > 2$,*
(3) $m = 3$ *and n is even, in which case $k = 5$, or*
(4) $m = n = 4$, *in which case $k = 4$.*

Thus, in precisely these cases, $\mathcal{P}(m, n)$ has a palindromic Ehrhart h-vector. We can say more (Section 4):

Theorem 4. *If $\mathcal{P}(m, n)$ is Gorenstein then $\mathcal{P}(m, n)$ has a unimodal Ehrhart h-vector. If m and n are both even, then $\mathcal{P}_T(m, n)$ also has a unimodal Ehrhart h-vector.*

(The sequence $h_0, h_1, \ldots, h_k$ is **unimodal** if there exists j such that $h_0 \leq \cdots \leq h_{j-1} \leq h_j \geq h_{j+1} \geq \cdots \geq h_k$.) Proposition 9 in Section 2 gives the dimension of $\mathcal{P}(m, n)$, so coupled with this result, we have an infinite list of Gorenstein polytopes.

The second theme of this paper concerns domino tilings. A **domino tiling** of an $m \times n$ rectangular board is a configuration of dominos such that the entire board is covered and there are no overlaps. An excellent survey on general tilings can be found in [1]. There is an obvious correspondence between domino tilings and perfect matchings of $\mathcal{G}(m, n)$. A natural question is whether there is an analogue of magic labellings of sum t of $\mathcal{G}(m, n)$. One possible answer is given by domino stackings. A **domino stacking of height** t of an $m \times n$ rectangular board is a collection of t domino tilings piled on top of one another. Every such domino stacking gives a magic labelling of sum t of $\mathcal{G}(m, n)$ in a natural way. In Section 5 we establish a simple connection between magic labellings and domino stackings by showing that this natural map is surjective via some general results for lattice polytopes:

Proposition 5. *Every magic labelling of sum t of $\mathcal{G}(m, n)$ can be realized as a domino stacking of height t of an $m \times n$ rectangular board.*

We should remark that this result is a consequence of Hall's marriage theorem, but that it will also follow from properties of the perfect matching polytopes.

It was shown by Klarner and Pollack [15] that for $t = 1$ and fixed m (or, by symmetry, fixed n), the generating function $\sum_{n \geq 0} T(m, n, 1) z^n$ is a proper rational function in z. In Section 5, we slightly modify this proof to obtain the same result for general t.

Theorem 6. *For fixed m and t, the sequences $(T(m, n, t))_{n \geq 0}$ and $(T(m, n, 1)^t)_{n \geq 0}$ are given by a linear homogeneous recurrence relation.*

Finally, in Section 6, we give a geometric approach to domino tilings. Our main result in this section is a new, geometric proof of the following:

Theorem 7 (Propp [16])**.** *If $T(m, n, 1)$ counts the number of ways to tile an $m \times n$ rectangular board with 2×1 dominos, then the sequence $(T(m, n, 1))_{n \geq 0}$ is given by a linear recurrence relation, and furthermore, this recurrence relation satisfies the reciprocity relation*

$$T(m, n, 1) = (-1)^n \, T(m, -n - 2, 1)$$

if $m \equiv 2 \bmod 4$, and

$$T(m, n, 1) = T(m, -n - 2, 1)$$

otherwise.

A remark: we interpret negative arguments to a recurrence relation to mean that the formula for the recurrence relation is run backwards. It was shown by Propp that even if one uses nonminimal recurrence relations, these values at negative numbers are well defined.

2. Basic Properties of perfect matching polytopes

Let G be a graph. For $x \in \mathbf{R}^{E(G)}$ and $S \subseteq V(G)$, we denote by $\partial(S)$ the set of edges that are incident to exactly one vertex in S, and $x(\partial(S))$ is the sum of the weights of the edges in $\partial(S)$ in the weighted graph associated with x. The following theorem gives an inequality description for $\mathcal{P}$.

Theorem 8 (Edmonds [9])**.** *Let G be a graph with an even number of vertices. A point $x = (x_e : e \in E(G))$ lies in the perfect matching polytope of G if and only if*

(1) $x_e \geq 0$ *for all $e \in E(G)$,*
(2) $x(\partial(v)) = 1$ *for all $v \in V(G)$,*
(3) $x(\partial(S)) \geq 1$ *for all $S \subseteq V(G)$ of odd size.*

A complete characterization of graphs for which condition (3) is redundant is given in [8]. In particular, the condition is redundant for bipartite graphs.

Proof of Proposition 1. As we just remarked, condition (3) in Theorem 8 is redundant for bipartite graphs. Thus $L_{\mathcal{P}}(t)$ counts the integer points $x = (x_e : e \in E(G))$ that satisfy

(1) $x_e \geq 0$ for all $e \in E(G)$,
(2) $x(\partial(v)) = t$ for all $v \in V(G)$,

whereas the interior points in $\mathcal{P}$ are those points satisfying (1) with strict inequality. In both cases, each equation in (2) involves exactly k variables, from which we deduce that $L_{\mathcal{P}^\circ}(t) = 0$ if $1 \leq t < k$ and $L_{\mathcal{P}^\circ}(t) = L_{\mathcal{P}}(t - k)$ for $t \geq k$. $\square$

An important piece of data associated with a polytope is its dimension. Fortunately, for grid graphs there is a simple formula for the dimension of its perfect matching polytope. The situation for torus graphs is complicated only in the sense that there are several cases to consider.

Proposition 9. *Suppose that mn is even.*

(1) $\dim \mathcal{P}(m, n) = (m - 1)(n - 1)$.
(2) *If $n > 2$ is even, then $\dim \mathcal{P}_T(1, n) = 1$.*
(3) *If $n > 2$ is even, then $\dim \mathcal{P}_T(2, n) = n + 1$.*
(4) *If $n > 1$ is odd, then $\dim \mathcal{P}_T(2, n) = n$.*
(5) *If $m > 2$ and $n > 2$ are both even, then $\dim \mathcal{P}_T(m, n) = mn + 1$.*
(6) *If $m > 2$ is even and $n > 1$ is odd, then $\dim \mathcal{P}_T(m, n) = mn$.*

The proof of this statement follows easily from Theorem 10 below. But we first need some definitions. A graph G is said to be **matching covered** if every edge of G belongs to a perfect matching of G. A **brick** is a 3-connected and bicritical graph (i.e., removing any two vertices results in a connected graph that has a perfect matching). A method for decomposing general graphs into bricks is given in [10], and the number of such bricks obtained is denoted by $B(G)$. We shall not need this machinery; for our purposes it is enough to know that $B(G) = 0$ if G is bipartite, and that $B(G) = 1$ if G is a brick. There is no overlap since the definition of a brick prevents it from being bipartite.

Theorem 10 (Edmonds–Lovász–Pulleyblank [10])**.** *Let G be a matching covered graph. If $\mathcal{P}$ is the perfect matching polytope of G, then $\dim \mathcal{P} = \#(E(G)) - \#(V(G)) + 1 - B(G)$.*

Proof of Proposition 9. Both $\mathcal{G}(m, n)$ and $\mathcal{G}_T(m, n)$ are matching covered if $m > 1$ and $n > 1$. The number of edges of $\mathcal{G}(m, n)$ is $m(n - 1) + n(m - 1)$, and the number of vertices is mn. In the case that either $m = 1$ or $n = 1$, $\mathcal{P}(m, n)$ consists of a single point, which has dimension 0 and agrees with the formula.

When m and n are both even, $\mathcal{G}_T(m, n)$ is bipartite, so it is enough to count edges. For $m = 2$, the number of edges is $3n$, and when $m > 2$, the number of edges is $2mn$.

In the other cases when m is even and $n > 1$ is odd, $\mathcal{G}_T(m, n)$ is not bipartite, but is 3-connected and bicritical if $n > 1$. So in this case, $\dim \mathcal{P}_T(m, n) = \#(E(G)) - \#(V(G))$, and we just need to count edges, which we've done above for $n > 1$. For the case $m > 2$ and $n = 1$, there are exactly two perfect matchings of $\mathcal{G}_T(m, 1)$, which gives $\dim \mathcal{P}_T(m, 1) = 1$. $\square$

3. Characterization of Grid Graphs with Gorenstein Polytopes

Now we give a lemma that will be used in the proof of Theorem 3.

Lemma 11. *Let $n > 2$ be even. Given a magic labelling of sum t of $\mathcal{G}(3, n)$, let a_i, b_i, and c_i denote the values of the ith edge in the top, middle, and bottom rows, respectively. Then for i even, $c_i = b_i - a_i$ and for i odd, $c_i = t - a_i + b_i$.*
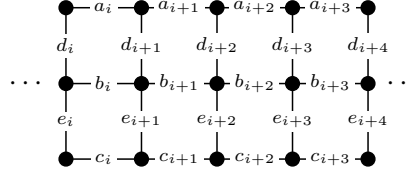
FIGURE 1. Proof of Lemma 11.

Proof. We refer to Figure 1. Here we are assuming that i is even and $1 < i < n - 4$. Following the constraints given by the vertices, we get $d_{i+2} = t - a_{i+1} - a_{i+2}$, which implies

$$e_{i+2} = t - (t - a_{i+1} - a_{i+2}) - b_{i+1} - b_{i+2}$$
$$= a_{i+1} + a_{i+2} - b_{i+1} - b_{i+2}.$$

By induction, $c_{i+1} = t - a_{i+1} + b_{i+1}$, so

$$c_{i+2} = t - (t - a_{i+1} + b_{i+1}) - (a_{i+1} + a_{i+2} - b_{i+1} - b_{i+2})$$
$$= b_{i+2} - a_{i+2}.$$

We play the same game for c_{i+3}. Solving some equations gives $d_{i+3} = t - a_{i+2} - a_{i+3}$ and

$$e_{i+3} = t - (t - a_{i+2} - a_{i+3}) - b_{i+2} - b_{i+3}$$
$$= a_{i+2} + a_{i+3} - b_{i+2} - b_{i+3}.$$

Using this identity, we finally conclude

$$c_{i+3} = t - c_{i+2} - e_{i+3}$$
$$= t - (b_{i+2} - a_{i+2}) - (a_{i+2} + a_{i+3} - b_{i+2} - b_{i+3})$$
$$= t - a_{i+3} + b_{i+3}.$$

However, we have already taken care of the base case, as setting variables with negative index to be 0 is equivalent to not having them there at all, and the proof proceeds in the same way. □
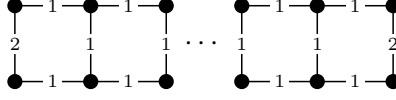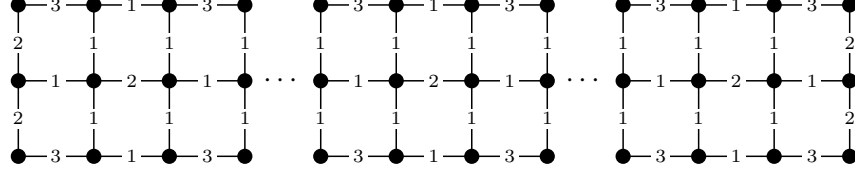
Now we turn to the proof of Theorem 3. Recall its statement: $\mathcal{P} = \mathcal{P}(m, n)$ is Gorenstein if and only if one of the following holds:

(1) $m = 1$ and n is even,
(2) $m = 2$,
(3) $m = 3$ and n is even, or
(4) $m = n = 4$.

Proof of Theorem 3. In order to show that a polytope $\mathcal{P}$ is Gorenstein of index k, we prove two things. First, we show that $k\mathcal{P}$ is the smallest dilate of $\mathcal{P}$ to contain an interior integer point, and that this point is unique. Second, we show that if p is the unique integer point in $k\mathcal{P}^\circ$, then for any integer point $x \in t\mathcal{P}^\circ$, $x_i \geq p_i$. If our graph is bipartite, then this is enough to conclude that $x - p \in (t - k)\mathcal{P}$ using the hyperplane description given by Theorem 8. This gives an obvious bijection which implies that $L_{\mathcal{P}^\circ}(t) = L_{\mathcal{P}}(t - k)$.

If $m = 1$ and n is even, then there is exactly one matching of $\mathcal{G}(m, n)$. Then $\mathcal{P}$ is a point, so the Ehrhart series is $\text{Ehr}_{\mathcal{P}}(z) = \frac{1}{1-z}$, and the statement follows.

We now consider the case $m = 2$. If $n = 2$, $\mathcal{P}$ is a 2-regular graph and Proposition 1 applies. For $n > 2$, note that the polytope $t\mathcal{P}$ has no interior points if $t < 3$. For $t = 3$, an interior point is given in Figure 2. The vertices of degree 3 force their adjacent edges to have weight 1, so it follows that this is the only such interior point. For $t \geq 3$, the edges of an interior point of $t\mathcal{P}$ must have weight ≥ 2 if they have weight 2 in the figure. If not, then their adjacent edges have weight $t - 1$

FIGURE 2. The interior point of $3\mathcal{P}$ for $\mathcal{G}(2, n)$.

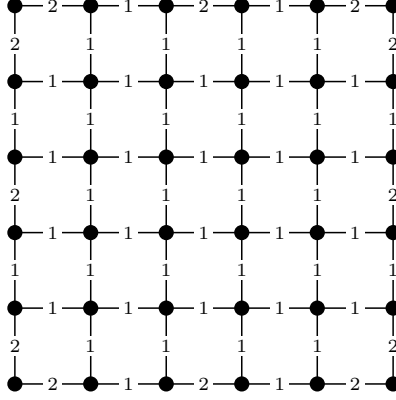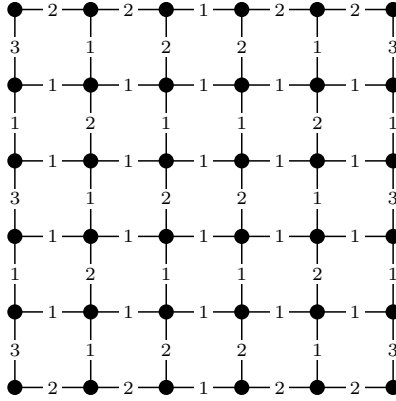FIGURE 3. The interior point of $5\mathcal{P}$ for $\mathcal{G}(3, n)$ and n even.

and they are adjacent to vertices of degree 3, which gives a contradiction. So $\mathcal{P}$ is Gorenstein in this case.

For the case $m = 3$, we may assume that $n > 2$. The existence of vertices of degree 4 means that $t\mathcal{P}$ has no interior points for $t < 4$. To get an interior point for $t = 4$, consider a vertex v of degree 2. If the edges adjacent to v have weight 1 and 3, then the edge of weight 3 is adjacent to a vertex of degree 3, which cannot happen. Thus, they must both have weight 2. However, there is a vertex w of degree 3 that is adjacent to two vertices of degree 2. Since the edges that they share both have weight 2, there is no valid weight for the third edge adjacent to w, so $4\mathcal{P}$ also has no interior points. Using the notation of Lemma 11, we get an interior point of $5\mathcal{P}$ with the following labelling: $a_i = c_i = 3$ if i is odd and $a_i = c_i = 1$ otherwise, $b_i = 1$ if i is odd and $b_i = 2$ if i is even, $d_i = e_i = 2$ if $i = 1$ or $i = n + 1$ and $d_i = e_i = 1$ otherwise. We illustrate this interior point in Figure 3. To see this is the only interior point of $5\mathcal{P}$, we consider an alternative labelling. No edge can have weight 4, so the weights for vertices of degree 2 must be 2 and 3. Note that the 3 and 2 must be assigned as they are in Figure 3. Otherwise, the vertex of degree 3 on the far left will have sum greater than 5. We will show that for $t \geq 5$, $a_i, c_i \geq 3$ if i is odd and that $b_i \geq 2$ if i is even, which implies the uniqueness of the given interior point. By Lemma 11, we know that $c_i = b_i - a_i$ if i is even and $c_i = t - a_i + b_i$ otherwise. For the first claim, suppose otherwise. Then $3 > t - a_i + b_i \geq t - a_i + 1$, which implies $a_i > t - 2$, or $a_i \geq t - 1$. However, a_i is adjacent to a vertex of degree 3 if $i > 1$, so this is a contradiction. For the second claim, if $b_1 = 1$, then $c_i = 1 - a_i$, which means $c_i \leq 0$, which cannot happen. Thus, we have shown uniqueness of the interior point in $5\mathcal{P}$. But we have done more. From the inequalities shown, we get for free the functional identity $L_{\mathcal{P}^\circ}(t) = L_{\mathcal{P}}(t - k)$, giving that $\mathcal{P}$ is Gorenstein.

When $m = n = 4$, there are no interior points for $t\mathcal{P}$ when $t \leq 3$, and for $t = 4$, a unique interior point is given by labelling the eight edges incident to vertices of degree 2 with weight 2 and the rest of the edges weight 1. For a general interior point in $t\mathcal{P}$ when $t \geq 4$, the edges adjacent to the corner edges must have at least weight 2, so $\mathcal{P}$ is Gorenstein.

For the other (m, n) of interest, we split them up into two cases. In the first case, $m \geq 4$ and $n > 4$ are both even, and in the second case, $m \geq 4$ is even and $n > 4$ is odd.

First suppose that $m \geq 4$ and $n > 4$ are both even. There are no interior points in $t\mathcal{P}$ for $t < 4$. We give an interior point for $4\mathcal{P}$ as follows. Assign weight 2 to the edges $\{(0, i), (0, i + 1)\}$, $\{(m - 1, i), (m - 1, i + 1)\}$, $\{(i, 0), (i + 1, 0)\}$, and $\{(i, n - 1), (i + 1, n - 1)\}$ when i is even, and assign 1 to all other edges. We illustrate this for $\mathcal{G}(6, 6)$ in Figure 4. This example is big enough to be instructive. To see this is unique, note that the weights of the edges of the corner vertices must be 2, and the weights of the edges incident to a vertex of degree 4 must be 1. This, however, forces the

FIGURE 4. The interior point of $4\mathcal{P}$ for $\mathcal{G}(6,6)$.

FIGURE 5. An interior point of $5\mathcal{P}$ for $\mathcal{G}(6,6)$.
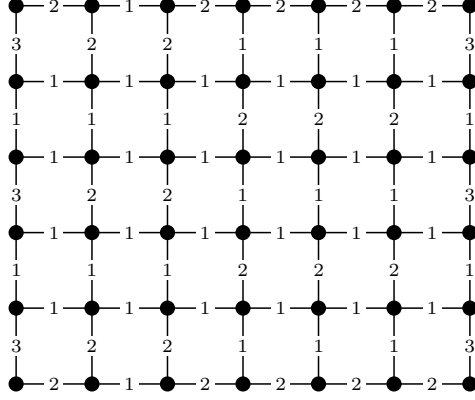
rest of the values on the edges incident to vertices of degree 3. Now consider the following interior point of $5\mathcal{P}$.

- Assign weight 2 to edges of the form $\{(i,0),(i,1)\}$, $\{(i,j),(i,j+1)\}$, and $\{(i,n-2),(i,n-1)\}$, for $i \in \{0, m-1\}$ and j odd.
- Assign weight 3 to edges of the form $\{(i,j),(i+1,j)\}$ for i even and $j \in \{0, n-1\}$.
- Assign weight 2 to edges of the form $\{(i,j),(i+1,j)\}$ for i odd and $j \in \{1, n-2\}$.
- Assign weight 2 to edges of the form $\{(i,j),(i+1,j)\}$ for i even and $2 \leq j \leq n-3$.
- Finally, assign weight 1 to all other edges.

We illustrate this for $\mathcal{G}(6,6)$ in Figure 5 (here $(0,0)$ is the bottom left vertex and $(5,5)$ is the top right vertex). In order for $\mathcal{P}$ to be Gorenstein, we must have $L_{\mathcal{P}^\circ}(t+4) = L_{\mathcal{P}}(t)$. Note that for any point of $\mathcal{P}$, we may add the interior point of $4\mathcal{P}$ to get an interior point of $5\mathcal{P}$. However, all points acquired through this must have weight ≥ 2 for the edge $\{(0,2),(0,3)\}$, but we have given an interior point of $5\mathcal{P}$ such that this edge has weight 1, which implies $L_{\mathcal{P}^\circ}(5) > L_{\mathcal{P}}(1)$, so the functional identity is not satisfied.

In the case that $m \geq 4$ is even and n is odd, $t\mathcal{P}$ does not have an interior point for $t \leq 4$. To see that this is true for $t = 4$, the edges incident to the corner vertices must have weight 2. This forces a labelling as in the case n is even, but it will not work because n is now odd. In other words, if we label the outside and work toward the middle, there will be no satisfactory values for the middle edges. We now give an interior point for $5\mathcal{P}$.

- Assign weight 3 to all edges of the form $\{(i,j),(i,j+1)\}$ where $i \in \{0, n-1\}$ and j is even.

FIGURE 6. An interior point of $5\mathcal{P}$ for $\mathcal{G}(6,7)$.

- Assign weight 2 to edges of the form $\{(i,j),(i+1,j)\}$ where $j \in \{0, m-1\}$ and $i \neq 1$.
- Assign weight 2 to edges $\{(i,j),(i,j+1)\}$ where $i \in \{1,2\}$ and j is even.
- Assign weight 2 to edges $\{(i,j),(i,j+1)\}$ where $3 \leq i \leq n-2$ and j is odd.
- Finally, assign weight 1 to all other edges.

This labelling is illustrated for $\mathcal{G}(6,7)$ in Figure 6 (again, $(0,0)$ is the bottom left vertex and $(5,6)$ is the top right). Since this graph labelling is not symmetric, we can get another interior point by "flipping" the graph (i.e., the weight for edges $\{(i,j),(i,j+1)\}$ and $\{(i,j),(i+1,j)\}$ are the weights for $\{(n-1-i,j),(n-1-i,j+1)\}$ and $\{(n-2-i,j),(n-1-i,j)\}$, respectively, from the interior point given previously). Thus, $5\mathcal{P}$ does not have a unique interior point so $\mathcal{P}$ is not Gorenstein. $\qquad\square$

We give an example of the Ehrhart polynomial and Ehrhart series of the perfect matching polytope of a grid graph. We used polymake [12] for conversion from the hyperplane description given in Theorem 8 to vertex descriptions, and normaliz [6] for the computation of the Ehrhart functions.

Example. The Ehrhart functions of $\mathcal{P}(3,4)$ are

$$L_{\mathcal{P}(3,4)}(t) = \frac{1}{120}(t+1)(t+2)(t+3)(t+4)(t^2+5t+5),$$

$$\mathrm{Ehr}_{\mathcal{P}(3,4)}(z) = \frac{z^2+4z+1}{(1-z)^7}.$$

The smallest example of a non-Gorenstein polytope arises from $\mathcal{P}(4,5)$, which has Ehrhart series

$$\mathrm{Ehr}_{\mathcal{P}(4,5)}(z) = \frac{21z^8 + 760z^7 + 5919z^6 + 15578z^5 + 16432z^4 + 7356z^3 + 1339z^2 + 82z + 1}{(1-z)^{13}}.$$

4. Unimodular Triangulations and Unimodality

We remind the reader that a **triangulation** of a polytope $\mathcal{P}$ is a set of simplices Δ whose union is $\mathcal{P}$ such that the intersection of any two simplices is a face of both. An integral simplex with vertices $v_0, \ldots, v_n$ is **unimodular** if $\{v_1 - v_0, \ldots, v_n - v_0\}$ forms a basis of $\mathbf{Z}^n$ as a free Abelian group, and a triangulation is unimodular if each simplex is unimodular.

One might ask if $\mathcal{P}(m,n)$ has a unimodular triangulation. Fortunately, the answer is yes. Given an ordering $\tau = (v_1, \ldots, v_n)$ of the vertices of a polytope $\mathcal{P}$, the **reverse lexicographic triangulation** Δ_τ is defined as follows. For a single vertex v, $\Delta_\tau = \{v\}$. In general, consider all facets (codimension 1 faces) of $\mathcal{P}$ that do not contain v_n. For each such facet F, there is an ordering

of the vertices of F induced by τ that gives a reverse lexicographic triangulation of F. For each simplex in this triangulation, take the convex hull of its union with v_n. The union of all such convex hulls is Δ_τ. A polytope is said to be **compressed** if the reverse lexicographic triangulation with respect to any ordering of its vertices is unimodular. The following result immediately guarantees the existence of unimodular triangulations.

Theorem 12 (Santos [18], Ohsugi-Hibi [13], Sullivant [23]). *If $\mathcal{P}$ is a 0/1 polytope (i.e., each coordinate of each vertex is either 0 or 1) defined by the linear equalities and inequalities*

$$\mathbf{A}\,\mathbf{x} = \mathbf{u} \quad and \quad 0 \le x_i \le 1,$$

where $\mathbf{A}$ is an integer-valued matrix and $\mathbf{u}$ is an integer-valued column vector, then $\mathcal{P}$ is compressed.

As a bonus, we obtain Theorem 4, that is, for the (m, n) discussed in Theorem 3, the Ehrhart h-vector is unimodal. This is an immediate consequence of Theorem 12 combined with the following result.

Theorem 13 (Athanasiadis–Bruns–Römer [2, 7]). *If $\mathcal{P}$ is compressed and Gorenstein, then the Ehrhart h-vector of $\mathcal{P}$ is unimodal.*

5. Domino Stackings and Magic Labellings

There is a natural many-to-one function from domino stackings of height t of an $m \times n$ rectangular board to the magic labellings of sum t of $\mathcal{G}(m, n)$. It is conceivable that some magic labelling cannot be realized as a physical stacking. However, we have shown earlier that $\mathcal{P}(m, n)$ has a unimodular triangulation, which implies that it is a normal polytope. (An integral polytope $\mathcal{P}$ is **normal** if every integer point in $t\mathcal{P}$ can be written as a sum of t integer points in $\mathcal{P}$.) In particular, this means that every magic labelling of sum t of $\mathcal{G}(m, n)$ must be the sum of t perfect matchings, so our function is a surjection, and this is what we claimed in Proposition 5.

Recall that for fixed m, the generating function $\sum_{n \ge 0} T(m, n, 1)z^n$ evaluates to a proper rational function in z, which means that the sequence $(T(m, n, 1))_{n \ge 0}$ is given by a linear recurrence relation. Running these recurrence relations backwards, one obtains values for $T(m, n, 1)$ when $n < 0$. It was shown in [16] that these values for $n < 0$ are the same even if the recurrence relation used is not minimal. A reciprocity relation was also given, which is the content of Theorem 7.

This is equivalent to a result in [22] which gives properties of the numerator and denominator polynomials in the rational function $\sum_{n \ge 0} T(m, n, 1)z^t$. We would like an analogous result for magic labellings of sum t and domino stackings of height t. Recall that Theorem 6 states that for general t, if one fixes m, then there is a recurrence relation in n for the function $T(m, n, t)$. The proof that we give is essentially the same as the one found in [15] with only slight modifications.

Proof of Theorem 6. Given a magic labelling of sum t for $\mathcal{G}(m, n)$, consider its canonical planar drawing. We can uniquely encode it with $n + 1$ vectors $c_0, \ldots, c_n$ of length m in the following way. For $1 \le i \le n - 1$, the jth entry of c_i is the label of the jth horizontal edge counting from the top of the graph in the ith column starting from the left and going right. We define c_0 and c_n to be zero vectors of length m.

We now form a directed graph $\mathscr{G}_m$ whose vertex set is $\{0, 1, \ldots, t\}^m$. For two vectors $u, v \in \{0, \ldots, t\}^m$, we form the edge (u, v) if and only if there is some n such that there is a magic labelling of sum t for $\mathcal{G}(m, n)$ encoded by the vectors $c_0, \ldots, c_n$ with $c_i = u$ and $c_{i+1} = v$ for some i. Given this, there is a natural bijection between the magic labellings of sum t of $\mathcal{G}(m, n)$ and walks of length n in $\mathscr{G}_m$ from $\{0\}^m$ to itself.

Order the vertices of $\mathscr{G}_m$ by interpreting each vector as a number in base $t + 1$, and let A_m be the adjacency matrix of $\mathscr{G}_m$. The number of walks of length n from $\{0\}^m$ to itself is the $(0, 0)$ entry

of A_m^n. By the Cayley–Hamilton theorem, there is a polynomial relation

$$A_m^d + b_{d-1} A_m^{d-1} + \cdots + b_1 A_m + b_0 = 0$$

for $d = (t+1)^m$ and integers b_i. Multiplying both sides by A_m^{n-d} and looking at the $(0,0)$ entry in each term gives a recurrence relation for $T(m, n, t)$ involving the coefficients b_i.

In the case of domino stackings of height t, we can get a similar result. Instead of vectors in $\{0, \ldots, t\}^k$ making up the vertices of $\mathscr{G}_m$, we encode each domino stacking as a $k \times t$ zero-one matrix and play the same game to get a recurrence relation for the sequence $(T(m, n, 1)^t)_{n \geq 0}$. $\square$

As an immediate corollary, we now know that for fixed m and t, the generating function $\sum_{n \geq 0} T(m, n, t) z^n$ evaluates to a rational function $\frac{P(z)}{Q(z)}$. The next step is to understand what properties P and Q have.

The sequence $(T(2, n, 1))_{n \geq 0}$ is a shift of the Fibonacci numbers, so this gives a nice proof that powers (for a fixed exponent) of Fibonacci numbers can be encoded by a linear homogeneous recurrence relation. This was previously done in [17], which also gives recurrence relations for the *coefficients* of the recurrence relations.

6. Convex Geometric Considerations of Domino Tiling Reciprocity

In this section, we give a convex geometric proof of why reciprocity exists for the number of domino tilings of the $m \times n$ rectangular board for fixed m using the following theorem which gives a closed form. We make some remarks on reciprocity of domino stackings of fixed height t and width m at the end of the proof.

Theorem 14 (Kasteleyn [14]). *The number of domino tilings of the $m \times n$ rectangular board is given by*

$$T(m, n, 1) = \prod_{j=1}^{\lceil m/2 \rceil} \frac{c_j^{n+1} - \bar{c}_j^{n+1}}{c_j - \bar{c}_j}$$

where

$$c_j = \cos \frac{j\pi}{m+1} + \sqrt{1 + \cos^2 \frac{j\pi}{m+1}},$$

$$\bar{c}_j = \cos \frac{j\pi}{m+1} - \sqrt{1 + \cos^2 \frac{j\pi}{m+1}}.$$

We remark here that the above theorem is in a different form from the one found in [14]. The original formulation gives a product from $j = 1$ to $j = \lfloor m/2 \rfloor$, and designates $T(m, n, 1) = 0$ when mn is odd. However, when mn is odd, $c_j^{n+1} - \bar{c}_j^{n+1} = 0$ for $j = \lceil m/2 \rceil$, and for m odd and n even, it is $c_j - \bar{c}_j$, so our formulation is equivalent.

First note that $c_j \bar{c}_j = -1$ for all j. The key observation is that

$$\frac{c_j^{n+1} - \bar{c}_j^{n+1}}{c_j - \bar{c}_j} = c_j^n + c_j^{n-1} \bar{c}_j + \cdots + c_j \bar{c}_j^{n-1} + \bar{c}_j^n,$$

and that the sum on the right is an evaluation of a certain multivariate generating function $(*)$ below. That is, consider the polytope $\mathcal{P} = \mathrm{conv}\{(1,0), (0,1)\}$. Let $\ell := \lceil m/2 \rceil$; we are interested in the ℓ-fold Cartesian product of $\mathcal{P}$ with itself. Let $\mathcal{C} \subseteq \mathbf{R}^{2\ell+1}$ be the cone over $\mathcal{P}$. To be precise, $\mathcal{C} = \{(x, \lambda) \in \mathbf{R}^\ell \times \mathbf{R} : \lambda \geq 0, \ x \in \lambda P\}$. We consider the **integer point transform** of $\mathcal{C}$, which is the following formal Laurent series in $2\ell + 1$ variables.

$$(*) \qquad \sigma_{\mathcal{C}}(\mathbf{x}) := \sum_{\mathbf{y} \in \mathcal{C} \cap \mathbf{Z}^{2\ell+1}} \mathbf{x}^{\mathbf{y}},$$

where $\mathbf{x}^{\mathbf{y}} := x_1^{y_1} x_2^{y_2} \cdots x_{2\ell+1}^{y_{2\ell+1}}$. In the case of a simplicial cone $\mathcal{C}'$, if $y_1, \ldots, y_k$ are the generators of $\mathcal{C}'$, define the **fundamental parallelepiped**

$$\Pi(\mathcal{C}') := \{\lambda_1 y_1 + \cdots + \lambda_k y_k : 0 \le \lambda_i < 1\}.$$

Then the integer point transform $\sigma_{\mathcal{C}'}(\mathbf{x})$ is a rational function of the form (see, e.g., [4, Chapter 3])

$$\sigma_{\mathcal{C}'}(\mathbf{x}) = \frac{\sigma_{\Pi(\mathcal{C}')}(\mathbf{x})}{(1 - \mathbf{x}^{y_1}) \cdots (1 - \mathbf{x}^{y_k})}.$$

We can triangulate $\mathcal{P}$ (using no new vertices), and this triangulation induces a triangulation of $\mathcal{C}$ into simplicial cones. The generators of each will be of the form $x_{i_1} x_{i_2} \cdots x_{i_\ell}$ where $\{i_1, i_2, \ldots, i_\ell\} \subseteq \{1, \ldots, 2\ell\}$ contains exactly one of $2i-1$ and $2i$ for each $i = 1, \ldots, \ell$. Conversely, each such subset will appear as the generator for at least one of the simplicial cones in the triangulation of $\mathcal{C}$. By adding the rational functions for these simplicial cones and playing inclusion-exclusion on the intersections, the integer point transform of $\mathcal{C}$ is

$$\sigma_{\mathcal{C}}(\mathbf{x}) = \frac{P(\mathbf{x})}{\prod(1 - x_{i_1} x_{i_2} \cdots x_{i_\ell})}$$

where the product runs over all ℓ-subsets of $\{1, \ldots, 2\ell\}$ that contain exactly one of $2i-1$ and $2i$ for each $i = 1, \ldots, \ell$, and P is some multivariate polynomial. We make the substitutions $x_{2i-1} \mapsto c_i$, $x_{2i} \mapsto \bar{c}_i$, and $x_{2\ell+1} \mapsto z$. The polytope $n\mathcal{P}$ can be described as the set of nonnegative solutions to the equations $x_1 + \cdots + x_{2\ell} = \ell n$ and $x_{2i-1} + x_{2i} = n$ for each $i = 1, \ldots, \ell$. Thus, using Theorem 14,

$$\sigma_{n\mathcal{P}}(c_1, \bar{c}_1, \ldots, c_\ell, \bar{c}_\ell, z) = z^n \prod_{j=1}^{\ell} (c_j^n + c_j^{n-1}\bar{c}_j + \cdots + c_j \bar{c}_j^{n-1} + \bar{c}_j^n)$$

$$= T(m, n, 1)z^n.$$

From this substitution,

$$\sigma_{\mathcal{C}}(c_1, \bar{c}_1, \ldots, c_\ell, \bar{c}_\ell, z) = \sum_{n \ge 0} T(m, n, 1)z^n$$

$$= \frac{P(c_1, \bar{c}_1, \ldots, c_\ell, \bar{c}_\ell, z)}{\prod_I (1 - d_1^{i_1} d_2^{i_2} \cdots d_\ell^{i_\ell} z)},$$

where I runs over all possible binary strings of length ℓ, and i_j denotes the jth bit of I, and $d_j^0 = c_j$ and $d_j^1 = \bar{c}_j$.

To proceed, we use the following theorem due to Stanley which gives a relation between the integer point transform of a rational cone and the integer point transform of its interior.

Theorem 15 (Stanley [19]). *If $\mathcal{C}$ is a rational cone in $\mathbf{R}^d$ whose apex is the origin, then as rational functions,*

$$\sigma_{\mathcal{C}}(z_1^{-1}, \ldots, z_d^{-1}) = (-1)^{\dim \mathcal{C}} \sigma_{\mathcal{C}^\circ}(z_1, \ldots, z_d),$$

where $\mathcal{C}^\circ$ denotes the relative interior of $\mathcal{C}$.

Equipped with this fact, we have

$$
\begin{aligned}
\sigma_{\mathcal{C}^\circ}(c_1, \bar{c}_1, \ldots, c_\ell, \bar{c}_\ell, z) &= (-1)^{\ell+1} \sigma_{\mathcal{C}}(-\bar{c}_1, -c_1, \ldots, -\bar{c}_\ell, -c_\ell, z^{-1}) \\
&= (-1)^{\ell+1} \sigma_{\mathcal{C}}(c_1, \bar{c}_1, \ldots, c_\ell, \bar{c}_\ell, (-1)^\ell z^{-1}) \\
&= (-1)^{\ell+1} \sum_{n \leq 0} (-1)^{\ell n} T(m, -n, 1) z^n \\
&= (-1)^\ell \sum_{n > 0} (-1)^{\ell n} T(m, -n, 1) z^n.
\end{aligned}
$$

(1)

The last equality follows from the identity (of rational functions)

$$
\sum_{n \geq 0} f(n) z^n + \sum_{n < 0} f(n) z^n = 0,
$$

where $f(n)$ is given by a linear recurrence relation. Finally, we use the fact that $\mathcal{C}$ is a Gorenstein cone (i.e., adding some vector to the cone allows us to enumerate the integer points in its interior). This means that

$$
\begin{aligned}
\sigma_{\mathcal{C}^\circ}(c_1, \bar{c}_1, \ldots, c_\ell, \bar{c}_\ell, z) &= c_1 \bar{c}_1 \cdots c_\ell \bar{c}_\ell z^2 \sigma_{\mathcal{C}}(c_1, \bar{c}_1, \ldots, c_\ell, \bar{c}_\ell, z) \\
&= (-1)^\ell z^2 \sum_{n \geq 0} T(m, n, 1) z^n.
\end{aligned}
$$

(2)

Putting both sets of equalities together and matching coefficients gives

$$
T(m, n, 1) = (-1)^{\ell n}\, T(m, -n - 2, 1).
$$

To recover the result in [16], note that when $m \equiv 3 \pmod 4$ or $m \equiv 0 \pmod 4$, ℓ is even, so

$$
T(m, n, 1) = T(m, -n - 2, 1).
$$

In the other cases, ℓ is odd, so

$$
T(m, n, 1) = (-1)^n\, T(m, -n - 2, 1).
$$

In particular, for $m \equiv 1 \pmod 4$, $T(m, n, 1) = 0$ when n is odd, so this is equivalent to

$$
T(m, n, 1) = T(m, -n - 2, 1), \quad m \equiv 1 \pmod 4,
$$

which gives the desired reciprocity relations.

For general t, raise both sides of the reciprocity equations to the tth power to get reciprocity relations for domino stackings of height t. Alternatively, we could run through the above proof with the following small modifications. Consider the t-fold product of $\mathcal{P}$ with itself. The cone over $\mathcal{P}^t$ is now $(t\ell + 1)$-dimensional, so the last term in equation (1) becomes

$$
(-1)^{t\ell} \sum_{n > 0} (-1)^{t\ell n}\, T(m, -n, 1)^t z^n,
$$

and the last term in equation (2) becomes

$$
(-1)^{t\ell} z^2 \sum_{n \geq 0} T(m, n, 1)^t z^n.
$$

Combining these two gives the general reciprocity relation

$$
T(m, n, 1)^t = (-1)^{t\ell n}\, T(m, -n - 2, 1)^t.
$$

7. Further Directions

A natural question to ask is if unimodality of the Ehrhart h-vector extends to the (m, n) such that $\mathcal{P}(m, n)$ is not Gorenstein. Unfortunately, computing examples larger than $\mathcal{G}(4, 5)$ becomes difficult, so there is not much evidence supporting whether they are or are not unimodal.

We defined $\mathcal{G}(m, n)$ having vertex set $\{(i, j) \in \mathbf{Z}^2 : 0 \le i < m,\ 0 \le j < n\}$ with $\{(i, j), (i', j')\}$ an edge if and only if $|i - i'| + |j - j'| = 1$, but there is no reason to restrict to 2-dimensional lattices. We can define k-dimensional grid graphs $\mathcal{G}(m_1, m_2, \ldots, m_k)$ having vertex set $\{(x_1, \ldots, x_k) \in \mathbf{Z}^k : 0 \le x_i < m_i\}$ such that $\{(x_i), (y_i)\}$ is an edge if and only if $\sum_{i=1}^{k} |x_i - y_i| = 1$. What further things can one say about these higher dimensional analogues? It would make sense that very few of these graphs have Gorenstein polytopes. The perfect matching polytope of the cube graph $\mathcal{G}(2, 2, \ldots, 2)$ is Gorenstein for arbitrary k, since this is a k-regular graph. Some questions one can ask is if there is a largest k such that $\mathcal{G}(2, 2, \ldots, 2)$ is the only graph with a Gorenstein polytope (assuming that $m_i > 1$). If not, is there a largest k such that the number of nontrivial Gorenstein polytopes $\mathcal{P}(m_1, \ldots, m_k)$ is infinite? We can also ask about the function $T(m_1, \ldots, m_k; t)$ that counts the number of magic labellings of sum t of $\mathcal{G}(m_1, \ldots, m_k)$. Since multi-dimensional grid graphs are bipartite, $\sum_{t \ge 0} T(m_1, \ldots, m_k; t) z^t$ evaluates to a rational function. It is not difficult to see using the proof of Theorem 6 that $\sum_{m_i \ge 0} T(m_1, \ldots, m_k; t) z^{m_i}$ is also a rational function.

One could also define two domino stackings to be equivalent if the layers of tilings of one is a permutation of the layers of tilings of the other. Is there a recurrence relation for this new set of stackings? New techniques would be needed as the adjacency matrix argument no longer works.

As mentioned earlier, Riordan gives recurrence relations for coefficients of recurrence relations for the sequences given by powers of Fibonacci numbers in [17]. It is conceivable that the same phenomenon occurs for powers of $T(m, n, 1)$ for general m. Different techniques from what is presented here would most likely have to be employed. The main problem arises in that the proof of Theorem 6 does not give minimal recurrence relations. For instance, in the case $m = 2$ and $t = 1$, which is a shift of the Fibonacci sequence, a 5-term recurrence relation is given by the proof, but only a 3-term recurrence relation is needed. This corresponds to the fact that if this non-minimal recurrence relation appears in the denominator of the rational function $\sum_{n \ge 0} T(m, n, 1)^t z^n$, then it will not be in reduced terms. A further problem is that the adjacency matrix argument given above presents a recurrence relation that works for *all* of the entries of the matrix, but we are only interested in one specific entry.

Along those same lines, for fixed t, the sequence $T(m, n, t)$ for counting magic labellings of $\mathcal{G}(m, n)$ is given by a linear homogeneous recurrence relation if either m or n is fixed. It would be interesting to find a recurrence relation when m and n are both allowed to vary, and also to see if the coefficients that arise from these recurrence relations can also be shown to satisfy their own recurrence relation.

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